42nd International Mathematical Olympiad

Washington, DC, United States of America
July 8–9, 2001

Problems

Each problem is worth seven points.

Problem 1

Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} + \frac{b}{\sqrt{b^2 + 8\,c\,a}} + \frac{c}{\sqrt{c^2 + 8\,a\,b}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i\, a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Problem 5

In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$a\,c + b\,d = (b + d + a - c)\,(b + d - a + c).$$

Prove that $a\,b + c\,d$ is not prime.